**Nuclear Physics**

(8th lecture)

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  - The asymmetry energy
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  - Shells in atomic physics (reminder)
  - The nuclear single-particle shell model
  - The 1D and 3D harmonic oscillator case

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**The total kinetic energy of the nucleons:**

\[ E_k = \frac{1}{2(2\pi m)^2} \int \frac{p^2}{2M} n(p) d^3p = V \frac{2}{h^3} \int_0^{p_f} p^2 \cdot 4\pi p^2 dp = \frac{V}{h^3} \cdot \frac{4\pi}{M} \cdot \frac{p_f^5}{5} \]

**The total kinetic energy of the protons and the neutrons:**

\[ E_k^{(p)} = \frac{3}{10} \left( \frac{3\pi^2}{10} \right)^{3/2} \frac{h^2}{M} Z^{3/2} \]

\[ E_k^{(n)} = \frac{3}{10} \left( \frac{3\pi^2}{10} \right)^{3/2} \frac{h^2}{M} N^{3/2} \]

We note that \( \frac{N-Z}{A} \ll \frac{Z}{A} = \frac{1}{2} \), and a Taylor-expansion yields:

\[ Z^{3/2} = A^{3/2} \left( \frac{Z}{A} - \frac{N-Z}{A} \right)^{3/2} = A^{3/2} \left[ \left( \frac{1}{2} \right)^{3/2} - \frac{5}{3} \left( \frac{N-Z}{A} \right) \left( \frac{1}{2} \right)^{3/2} + \frac{5}{9} \left( \frac{N-Z}{A} \right)^2 \left( \frac{1}{2} \right)^{3/2} - \ldots \right] \]

\[ N^{3/2} = A^{3/2} \left( \frac{Z}{A} + \frac{N-Z}{A} \right)^{3/2} = A^{3/2} \left[ \left( \frac{1}{2} \right)^{3/2} + \frac{5}{3} \left( \frac{N-Z}{A} \right) \left( \frac{1}{2} \right)^{3/2} + \frac{5}{9} \left( \frac{N-Z}{A} \right)^2 \left( \frac{1}{2} \right)^{3/2} - \ldots \right] \]

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**The asymmetry energy:**

\[ A = \frac{N}{A} - \frac{Z}{A} \]

\[ A \equiv \frac{N-Z}{A} \]

Note: It has positive sign, therefore it decreases the binding energy!

(Unlike in the Weizsäcker formula)

In the second part only the second order terms remain (the linear terms cancel):

\[ A \left( \frac{N-Z}{A} \right)^2 = \left( \frac{N-Z}{A} \right)^2 \]

This way we get an asymmetry term, similar to the Weizsäcker formula’s!

Note: The asymmetry term also has a positive sign! This means that it increases the kinetic energy, and therefore decreases the binding energy (like in the Weizsäcker formula).
The **surface energy**:  
For simplicity consider now a cube (instead of a sphere)  
The wave function: \( \varphi(r) = \frac{2}{V} \sin(k_x x) \sin(k_y y) \sin(k_z z) \)  
where \( k_x = \frac{\pi}{L} n, \ k_y = \frac{\pi}{L} m, \ k_z = \frac{\pi}{L} \ell \) and \( k^2 = k_x^2 + k_y^2 + k_z^2 \)  
The number of states in the phase space \( V = L^3 \) :  
\[
\mathcal{N} = A = 2 - \frac{V}{(2\pi)^3} \frac{4\pi K^3}{3} = 2 - \frac{V}{8} \frac{4\pi}{3} \left( \frac{LK^3}{\pi} \right)
\]  
where \( K = k_f \)  
Volume of the 1/8 sphere in the space of \((n, m, \ell)\) quantum numbers  
So the total number of nucleons: \( A \propto K^3 \) from where: \( K \propto A^{1/3} \)  

The main assumption of the Fermi-gas model was, that the nucleons move „independently” – without interacting with each other – in an outside potential well. Is this not a crazy assumption for strongly interacting particles?  
**Answer:**  
\[
n(p) = \begin{cases} 1, & \text{if } p \leq p_f \\ 0, & \text{if } p > p_f \end{cases}
\]  
In the ground state of the nucleus every level is occupied up to the Fermi-level. Particles could scatter only out over the Fermi-level.  
If a particle gets higher energy and momentum (outside the Fermi-level) in a scattering process, energy and momentum conservation would mean that the „other” particle should get lower energy/momentum → no empty state there, **forbidden**!  
This shows also the **validity** of the Fermi-gas model: mainly the ground state properties.

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**Nuclear shell model**  
1) **Experimental indications**  
a) Neutron capture cross sections  
b) Binding energy difference of the last neutron
c) Excitation energy of the first excited states:

Conclusion: nuclei with the following Z and/or N numbers are extremely stable (in comparison with their neighbours): 2, 8, 20, 28, 50, 82, 126. These are the "magic numbers".

Nuclei with magic Z or N numbers are called "magic nuclei". If Z and N both are magic numbers, they are "double magic nuclei".

The following nuclei are double magic:

- $^2\text{He}$, $^{16}\text{O}$, $^{20}\text{Ca}$, $^{40}\text{Ca}$, $^{48}\text{Ni}$, $^{78}\text{Ni}$, $^{208}\text{Pb}$
- $^4\text{He}$, $^{18}\text{Ar}$, $^{36}\text{Kr}$, $^{54}\text{Xe}$, $^{86}\text{Rn}$

2) What characterizes a "shell"?

"Shell is a set of quantum-mechanical states with the same main quantum number". True?

Remember, in atomic physics:

- The noble gases are (experimental fact): magic numbers should be: 2, 10, 18, 36, 54, 86.
- The $n$th "shell" can hold theoretically $2n^2$ electrons. Therefore the theoretical magic numbers would be:
  - $n=1$: $2^2$ (☺)
  - $n=2$: $4^2$ (☺)
  - $n=3$: $6^2$ (??)
  - $n=4$: $8^2$ (??)
  - $n=5$: $10^2$ (??)

There is a discrepancy between theory and experiment!

Conclusion: The magic numbers in Nature are not defined by the rule above!

Shells in atomic physics (reminder):

H-atom (Bohr-model)

This determines the "energy-shells"!

$E = -\frac{me^2}{8\hbar^2 c^2} \frac{1}{r}$

$\rightarrow n = 1, 2, 3...$

The quantum-mechanical treatment (Schrödinger-equation):

$-\frac{\hbar^2}{2m} \nabla^2 + V(r) \psi(r, \theta, \phi) = E \cdot \psi(r, \theta, \phi)$ (spherical coordinates)

Wave function: $\psi_{n, l, m}(r, \theta, \phi) = \frac{R_n(r)}{r} Y_l^m(\theta, \phi) \rightarrow \begin{cases} n = 0, 1, 2, 3... \\ l = 0, 1, 2, 3... \\ -l \leq m \leq +l \end{cases}$

What is the relation between $n$ in the Bohr-model (defining the energy-shells), and the 3 quantum numbers of the wave-function?
### Shells in atomic physics (reminder):

**Bohr Quant. mech. No. of particles**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n )</th>
<th>( l )</th>
<th>( 2(2l+1) )</th>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1s 1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2s 2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2p 3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>3p 6</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0</td>
<td>3s 2</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2</td>
<td>4p 6</td>
</tr>
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<td>7</td>
<td>2</td>
<td>1</td>
<td>4d 10</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>0</td>
<td>4s 2</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>2</td>
<td>5d 10</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>3</td>
<td>5f 14</td>
</tr>
</tbody>
</table>

This determines the „shells“!

Why are the shells different in the Periodic System?

#### Why are the shells different in the Periodic System?

- **Bohr:***
  - The „size“ gets larger!

- **Why are the shells different in the Periodic System?***
  - The „screening“ of the occupied inner states deforms the potential!
  - Splitting of the states occur according to \( l \)!

**The shell parameters depend on the shape of the potential!**

### 3) The nuclear shell model

„The shell parameters depend on the shape of the potential!“

**How is the „nuclear potential“?***

The nuclear Hamiltonian:

\[
\hat{H} = \sum_{i=1}^{A} \left( -\frac{\hbar^2}{2m_i} \nabla_i^2 + \sum_{j \neq i} V_{ij}(\mathbf{r}_i, \mathbf{r}_j) \right)
\]

**A „trick“:**

\[
\hat{H} = \sum_{i=1}^{A} \left( -\frac{\hbar^2}{2m_i} \nabla_i^2 + \langle \mathbf{r}_i \rangle \right) + \left[ \sum_{i=1}^{A} \left( \langle \mathbf{r}_i \rangle \right) - \sum_{i=1}^{A} V(\mathbf{r}_i) \right]
\]

- **single-particle operators**
- **particle-particle interactions**

„mean“ potential

„residual“ interaction

(for single-particles)

(will be neglected)

This is the „single-particle shell model”

Every particle moves independently in a mean potential

### The nuclear shell model (contd.)

**The shape of the „mean“ nuclear potential.***

- **Central →** depends only on the absolute value of \( r \): \( V(r) \)
- **Similar shape as the nuclear density**
  - (since the nucleons create it)

It is difficult to solve the Schrödinger-equation for this shape.

Two different approximations:

- **square well**
- **harmonic oscillator**
The nuclear shell model (contd.)

Case #1: square well (infinite)!

\[ V(r) = \begin{cases} -V_0, & \text{if } r \leq R \\ \infty, & \text{if } r > R \end{cases} \quad (V_0 > 0) \]

Since the potential is central, the wave-function can be separated:

\[ \psi(r, \Omega) = \frac{\phi(r)}{r} \eta(\Omega) \]

The Schrödinger equation in the interior region in spherical coordinate system is:

\[ -\frac{\hbar^2}{2M} \Delta \psi(r, \Omega) - V \psi(r, \Omega) = E \psi(r, \Omega) \]

We get for the radial part:

\[ \frac{d^2 \phi_{n,l}}{dr^2} + \frac{2}{r} \frac{d \phi_{n,l}}{dr} + \left( \frac{2M}{\hbar^2} (V_0 - E) - \frac{l(l+1)}{r^2} \right) \phi_{n,l} = 0 \]

Solution: Bessel functions:

\[ \phi_{n,l}(r) = j_l(k_{n,l}r) \]

where \( k_{n,l}^2 = \frac{2M}{\hbar^2} (V_0 - E_{n,l}) \)

The functions have to go to 0 at \( r = R \)

\[ \Rightarrow \text{only roots of the Bessel functions are allowed} \to k_{n,l} \to \text{discrete values} \to E_{n,l} \to \text{discrete values too.} \]

<table>
<thead>
<tr>
<th>kR</th>
<th>(L = 0) (1s)</th>
<th>(2(L+1))</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi)</td>
<td>2</td>
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<td>2</td>
</tr>
<tr>
<td>4.49</td>
<td>L = 1 (2p)</td>
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<td>8</td>
</tr>
<tr>
<td>5.79</td>
<td>L = 2 (3d)</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td>2x</td>
<td>L = 0 (2s)</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>6.99</td>
<td>L = 3 (4f)</td>
<td>14</td>
<td>34</td>
</tr>
<tr>
<td>7.22</td>
<td>L = 1 (3p)</td>
<td>6</td>
<td>40</td>
</tr>
</tbody>
</table>

Second column: angular momentum quantum number \(L\) and the atomic physics notation of the state.

Note: inside a shell the higher \(L\) states go lower!

The nuclear shell model (contd.)

Case #2: the harmonic oscillator:

\[ V(r) = -V_0 + \frac{1}{2} m \omega^2 r^2 \]

Easiest solution is in Descartes coordinates

Wave function (in one dimension):

\[ \psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-m \omega x^2 / 2 \hbar} H_n \left( \sqrt{m \omega \hbar} \right) \]

where \( H_n(x) = (-1)^n e^{-x^2} \left( \frac{d}{dx} \right)^n e^{x^2} \) are the Hermite-polynomials

The energy spectrum:

\[ E_n = \hbar \omega \left( n + \frac{1}{2} \right) \quad n = 0, 1, 2, ... \]

The 3D harmonic oscillator (in Descartes coordinates):

\[ \psi_{n_x,n_y,n_z}(x,y,z) \propto e^{-\frac{m \omega (x^2 + y^2 + z^2)}{2 \hbar}} H_{n_x} \left( \sqrt{\frac{m \omega}{\hbar}} x \right) H_{n_y} \left( \sqrt{\frac{m \omega}{\hbar}} y \right) H_{n_z} \left( \sqrt{\frac{m \omega}{\hbar}} z \right) \]

\[ E_n = \hbar \omega \left( n_x + n_y + n_z + \frac{3}{2} \right) \quad \text{where} \quad n = n_x + n_y + n_z \]

The possible number of particles in a state:

\[ 2g = (n + 1)(n + 2) \]
The nuclear shell model (contd.)

The 3D harmonic oscillator (in spherical coordinates):
\[
\left( -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \psi(r, \vartheta, \varphi) = E \psi(r, \vartheta, \varphi)
\]

Wave function: \[\psi_{n,l,m}(r, \vartheta, \varphi) = \frac{R_n(r)}{r} Y_l^m(\vartheta, \varphi) \quad \begin{cases} n = 0,1,2,3... \\ l = 0,1,2,3... \\ -l \leq m \leq +l \end{cases} \]

Question:
What is the relation between \( n \) in the Descartes states (defining the energy-shells), and the 3 quantum numbers of the wave-function?

The idea for finding the correspondence: functions should be chosen to follow
- the same parity behaviour and
- the same number of degenerations!

The harmonic oscillator is a good „first“ approximation, but something more should be included!

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \pi )</th>
<th>((n + 1)(n + 2))</th>
<th>( n )</th>
<th>( l )</th>
<th>( 2(2l + 1) )</th>
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<td>4</td>
<td>5g</td>
</tr>
</tbody>
</table>

Remember:
\[ \psi_n(-r) = (-1)^n \psi_n(r) \]
\[ 2g = (n + 1)(n + 2) \]

Magic numbers:
2, 8, 20, 28, 50, 82, 126